Absorbing Boundary Conditions for anisotropic elastic waves

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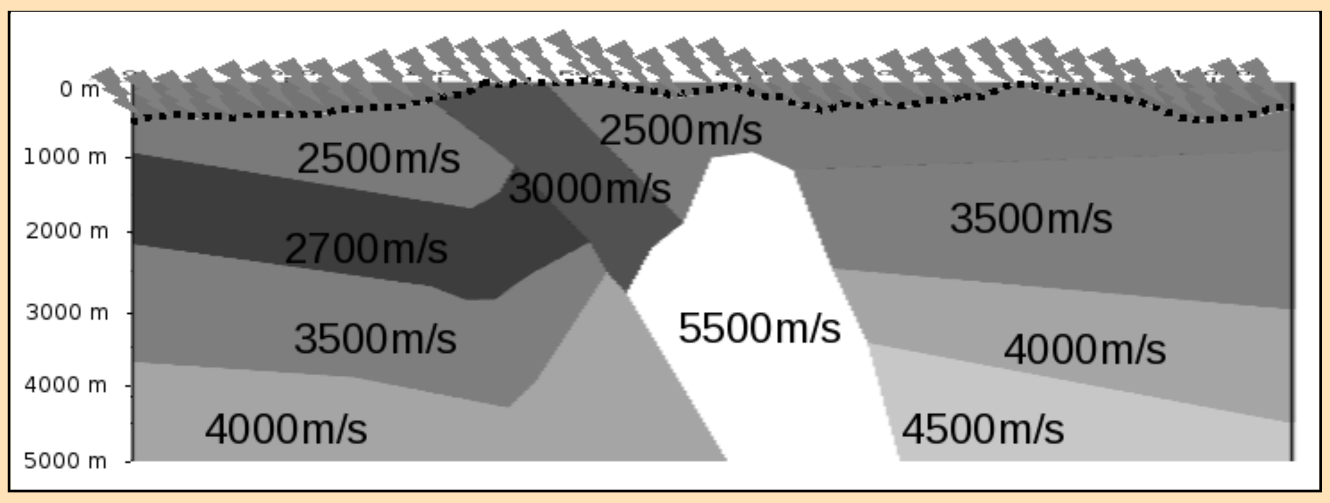
Depth Imaging Partnership (DIP), Inria-Total

Geophysics Simulation

Reverse Time Migration (RTM) is one of the most widely used technique of Seismic Imaging. It is based on successive solutions of the wave equation.



P-wave velocity model in a salt dome example

During the simulation, we cannot reproduce exactly the domain which is infinite in comparison with the wavelength. Thus, the computational domain is reduced to a box and efficient **boundary conditions** are applied so as to attenuate the reflexions.

It exists two common ways to do that: the Perfectly Matched Layers (PML), [1], which are easy to implement but instable with anisotropy; the Absorbing Boundary Conditions (ABC), [2], which are quite complex to implement with anisotropy.

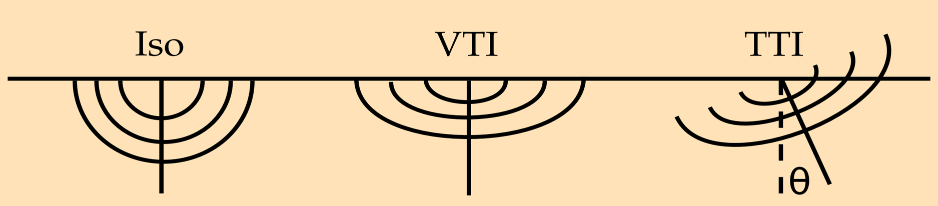
Anisotropic Elastodynamics

Denoting $\mathbf{x} = (x, z)$ and $t \geqslant 0$, the elastodynamics system reads as

$$\begin{cases} \rho(\mathbf{x})\partial_t \mathbf{v}(\mathbf{x}, t) &= \nabla.\underline{\underline{\sigma}}(\mathbf{x}, t), \\ \partial_t \underline{\underline{\sigma}}(\mathbf{x}, t) &= \underline{\underline{C}}(\mathbf{x}) : \underline{\underline{\epsilon}}(\mathbf{v}(\mathbf{x}, t)), \end{cases}$$

with $\rho > 0$ the density, $\mathbf{v}$ the velocity field, $\underline{\underline{\sigma}}$ the stress tensor, $\underline{\underline{C}}$ the stiffness tensor and $\underline{\underline{\epsilon}}$ the strain tensor with $\underline{\underline{\epsilon}}(\mathbf{v}) = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T)$.

Subsurface layers are assumed to be locally polar anisotropic, also called Transverse Isotropy (TI).
- Isotropic media are defined with the physical parameters ρ, V_p, V_s.
- Vertical TI (VTI) media use extra parameters ε, δ (and γ in 3D)
- Tilted TI (TTI) media use all of them plus an angle θ (and ϕ in 3D), see Thomsen [3].



Wavefronts for isotropic and TI media

Space & Time Discretization

The space discretization with the Discontinuous Galerkin Method (DGM) consists in approximating the functions $\mathbf{v}$ and $\underline{\underline{\sigma}}$ by discontinuous functions, regular enough on each mesh cell. The variational formulation reads as:

$$\begin{cases} \displaystyle\sum_K \int_K \rho_K \partial_t \mathbf{v}\mathbf{w} d\mathbf{x} = \sum_K \left(\int_{\Gamma_K} \{\{\underline{\underline{\sigma}}\mathbf{n}_K\}\}.[\![\mathbf{v}]\!] d\mathbf{x} - \int_K \underline{\underline{\sigma}} : \boldsymbol{\nabla}\mathbf{w} d\mathbf{x} \right), \\ \displaystyle\sum_K \int_K \partial_t \underline{\underline{\sigma}} : \underline{\underline{\xi}} d\mathbf{x} = \sum_K \left(\int_{\Gamma_K} [\![(\underline{\underline{C}} : \underline{\underline{\xi}})\mathbf{n}_K]\!]\{\{\mathbf{v}\}\} d\mathbf{x} - \int_K \mathbf{v}.\nabla.(\underline{\underline{C}} : \underline{\underline{\xi}}) d\mathbf{x} \right), \end{cases}$$

with the centered fluxes: $\{\{\mathbf{u}\}\} = (\mathbf{u}_{K_1} + \mathbf{u}_{K_2})/2$, $[\![\mathbf{u}]\!] = \mathbf{u}_{K_1} - \mathbf{u}_{K_2}$.
The time discretization is done with the classical Leap-Frog time scheme.

$$\begin{cases} \rho M_{\mathbf{v}} \dfrac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} + R_{\underline{\underline{\sigma}}} \underline{\underline{\sigma}}_h^{n+1/2} &= 0 \\ M_{\underline{\underline{\sigma}}} \dfrac{\underline{\underline{\sigma}}_h^{n+3/2} - \underline{\underline{\sigma}}_h^{n+1/2}}{\Delta t} + R_{\mathbf{v}} \mathbf{v}^{n+1} &= 0 \end{cases}$$

Since $M_{\mathbf{v}}$ and $M_{\underline{\underline{\sigma}}}$ are block-diagonal, this scheme is quasi-explicit.

Absorbing Boundary Conditions

The methodology for the construction of ABCs is based on the diagonalization of the elastodynamics system, see Enquist and Majda [1]. In practice, it can quickly become uneasy to use because of the coupling terms between P-waves and S-waves. A possible approach consists then in **uncoupling** these waves and constructing ABCs for each of them.

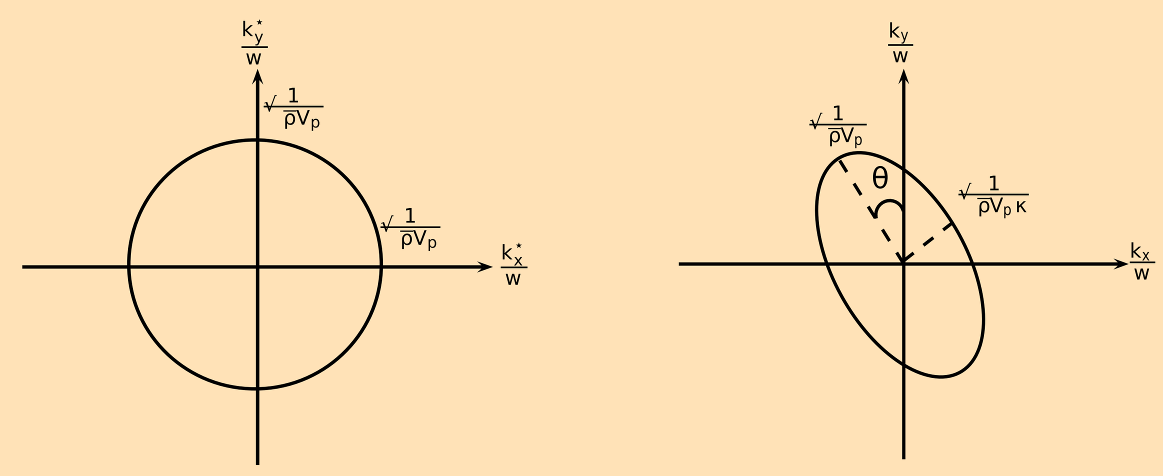
In case of a vertical boundary, the isotropic P-waves ABC reads as

$$\begin{cases} \sigma_{xx} &= \pm\sqrt{\rho} V_p v_x, \\ \sigma_{xz} &= 0. \end{cases}$$

And the isotropic S-waves ABC is

$$\begin{cases} \sigma_{xx} &= 0, \\ \sigma_{xz} &= \pm\sqrt{\rho} V_s v_z. \end{cases}$$

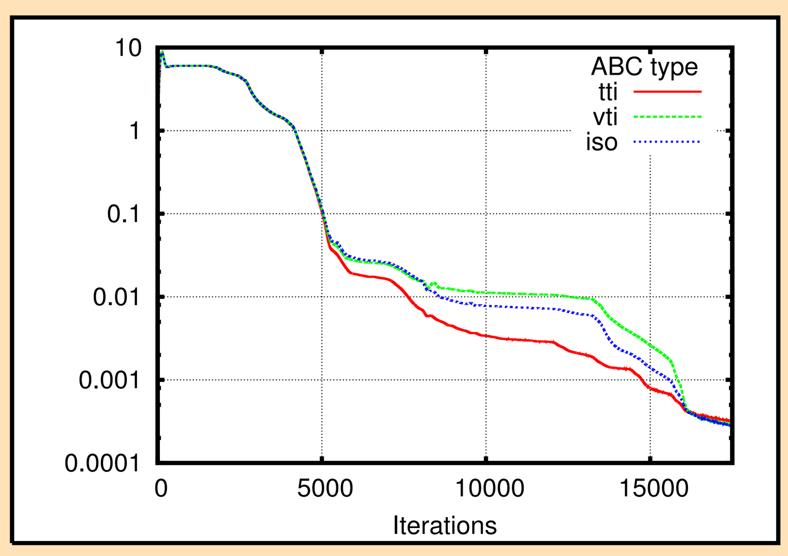
Elliptic anisotropy means that the TI coefficients are equal: $\delta = \varepsilon$. In this case, the slowness curve of S-waves forms a circle, as in the isotropic case, whereas the TTI P-waves slowness curve form a rotated ellipse .



P-wave curves in isotropic (left) and TTI (right) media

The construction of the new ABC is based on a **change of coordinate** which transforms a circle into a rotated ellipse. Introducing $\kappa = \sqrt{1 + 2\varepsilon}$, TTI P-waves ABC reads as

$$\begin{cases} \sigma_{xx} &= \pm\sqrt{\rho} V_p \frac{\kappa\cos^2\theta + \sin^2\theta}{(\kappa^2\cos^2\theta + \sin^2\theta)^{1/2}} \left[(\kappa\cos^2\theta + \sin^2\theta)v_x + (\kappa - 1)\cos\theta\sin\theta v_z\right], \\ \sigma_{xz} &= \pm\sqrt{\rho} V_p \frac{(\kappa - 1)\cos\theta\sin\theta}{(\cos^2\theta + \kappa^2\sin^2\theta)^{1/2}} \left[(1 - \kappa)\cos\theta\sin\theta v_x + (\cos^2\theta + \kappa\sin^2\theta)v_z\right]. \end{cases}$$



Discrete Energy

For a P-wave source, the figure shows three discrete energies measured in the same TTI medium, according to the type of ABC which is used. Our TTI ABC is the best, then the VTI and the isotropic ABCs lead to a worse absorption of about two-times higher.

Perspectives

A serious lead to obtain higher-order ABCs is the CRBC-LK idea (Complete Radiation Boundary Condition), which means an iterative boundary condition with a low-order ABC as final step.

RTM involves massively parallel computing. In this context, a comparison between ABC and PML would be interesting. The question is to know if ABC or PML are more adapted to supercomputing.

Bibliography

[1] Bérenger, J.-P. *A perfectly matched layer for the absorption of electromagnetic waves.* Journal of comptational physics, 1994, 114, 185-200

[2] B. Engquist and A. Majda *Absorbing boundary conditions for the numerical simulation of waves.* Math. Comp., 31:629-651, 1977

[3] L. Thomsen *Weak elastic anisotropy.* Geophy., 51(10):1954-1966, 1986

[4] S. Savadatti and M.N. Guddati *Accurate absorbing boundary conditions for anisotropic elastic media. part 2: Untilted non-elliptic anisotropy.* J. Comput. Phys., 2012.